Work and heat transfer in the presence of sliding friction

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The work done by frictional forces has usually been calculated incorrectly. The key to a correct treatment lies in making a careful distinction between a purely mechanical integral of Newton's second law on the one hand and the first law of thermodynamics on the other. These two equations are the same for point particles but differ for deformable systems, which include systems subject to sliding friction. A model-independent calculation is supplemented by applications to current models for friction. Heat transfer is treated in detail for the case of lubricated friction. Rotational friction is analyzed. An invariant form of the energy equation is presented.

I. A PARADOX

There is an apparent paradox in the traditional treatment of work and energy in the presence of sliding friction. Suppose a block is dragged at constant speed across a table with friction. The applied force f acting through a distance d does an amount of work fd. The frictional force μN is equal to f, since there is no acceleration. It would seem that the frictional force does an amount of work $-\mu Nd = -fd$. The total work would be $fd - fd = 0$. For a point particle, the work done is equal to the change in kinetic energy,

$$fd - fd = 0 = \Delta\left(\tfrac{1}{2}Mv^2\right),$$

which shows that the speed v does not change. This is consistent with there being no acceleration, so this all seems correct.

Yet there is something terribly wrong here. Where is the energy term representing the increased internal energy of the block? The block warms up due to friction, and the increase in thermal energy of the block can be measured by calorimetry. The simple treatment outlined above gives the right answer for the change in speed of the block, yet the work-energy equation does not contain a term representing a major energy change in the system. This paradox can be resolved, and a more detailed picture of friction emerges in the process. In Sec. VIII this new approach is applied to a typical homework problem involving friction, and it is shown that the traditional analysis found in most textbooks is incorrect.

II. THE CM EQUATION

The key to the paradox lies in the difference between the true energy equation, called for historical reasons the first law of thermodynamics, and the "pseudowork-energy" equation or "CM" (center-of-mass) equation.[1–3] The CM equation is a spatial integral of Newton's second law for a system, obtained by integrating through the displacement of the center of mass:

$$\sum_i \mathbf{F}_{i_{\text{external}}} = M\mathbf{a}_{\text{CM}}, \tag{1a}$$

$$\int\left(\sum_i \mathbf{F}_{i_{\text{external}}}\right)\cdot d\mathbf{r}_{\text{CM}} = M\int \frac{d\mathbf{v}_{\text{CM}}}{dt}\cdot d\mathbf{r}_{\text{CM}}, \tag{1b}$$

$$\int\left(\sum_i \mathbf{F}_{i_{\text{external}}}\right)\cdot d\mathbf{r}_{\text{CM}} = \Delta\left(\tfrac{1}{2}Mv_{\text{CM}}^2\right). \tag{1c}$$

This CM equation looks seductively like an energy equation, but it deals with center-of-mass quantities. It does not include rotational or vibrational energy, nor nonmechanical forms of energy such as thermal or chemical energy. This derivation can also be carried out just for the x, y, or z component of Newton's second law, yielding three separately valid equations whose algebraic sum is the CM equation. The separate validity of these three equations provides an additional proof that the CM equation is not really an energy equation, but is more closely related to momentum.

When a block is dragged at constant speed across a table, the distance the block moves is the displacement d_{CM} of the center-of-mass point, and the CM equation for the process is

$$(f - f)d_{\text{CM}} = 0 = \Delta\left(\tfrac{1}{2}Mv_{\text{CM}}^2\right). \tag{2}$$

This CM equation is equivalent to Newton's second law for the process, $(f - f) = 0 = Ma_{\text{CM}}$, which also shows that v_{CM} is constant. A start on resolving the paradox concerning the block is the recognition that what we originally thought was an energy equation was actually the CM equation.

Since it springs from the purely mechanical second law of Newton, the CM equation for the sliding block does not yield any information about the rise in thermal energy of the block, nor about possible heat transfer between the block and its surroundings. To describe these energy aspects of the process, we must invoke the first law of thermodynamics, hereafter abbreviated as the FLT. The CM and FLT equations together give a more complete picture of the process than either of them alone can give.

A highly symmetrical situation will be considered in detail. This will lead to a surprising conclusion which is totally independent of any particular model for sliding friction. Specific models of friction will then be used to illustrate how this unusual model-independent result comes about in practice.

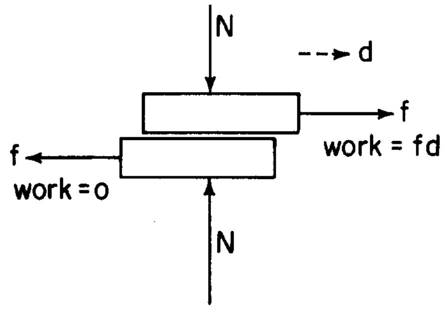

Fig. 1. Two identical blocks slide across each other with friction, at constant speed. Viewed from the inertial frame of the lower block, the force on the left does no work, and the force on the right does an amount of work fd. The vertical forces are applied by thermally insulated rollers, and the system is in deep space, away from significant gravitational forces.

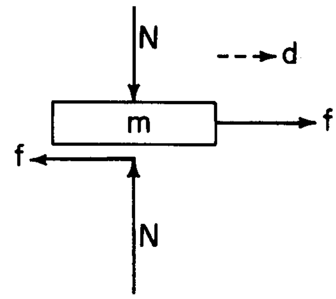

Fig. 2. The freebody diagram for the upper block alone. The force on the right does an amount of work fd, but the frictional force f to the left only does an amount of work $-fd_{\text{eff}}$. It turns out that d_{eff} is less than d.

III. A MODEL-INDEPENDENT CALCULATION

Consider a very simple and highly symmetrical situation. Two identical blocks are pulled *at constant speed* across each other (Fig. 1). For the sake of absolute symmetry, let the experiment be carried out in deep space away from gravity, with the normal forces N applied by thermally insulated rollers on the outer surfaces of the two blocks. Assume the total displacement is small compared to the length of the blocks (or equivalently that the blocks are very long), so that conditions are hardly changing (e.g., area of contact and amount of overhang). Alternatively, one could consider a ring rotating on a similar ring, to insure constant mechanical conditions.

We choose to observe the motion from a frame moving with the lower block, so that the lower block is stationary in our inertial reference frame. Take both blocks as the system to be analyzed. The center of mass moves a distance $d/2$ as the upper block moves a distance d. Since the lower block does not move, the force on the lower block does no work, but it does appear in the CM equation:

$$\text{CM:}\quad (f-f)(d/2) = \Delta\left[\tfrac{1}{2}(2m)v_{\text{CM}}^2\right] = 0, \tag{3a}$$

$$\text{FLT:}\quad fd = \Delta E_{\text{thermal, upper and lower blocks}}\,. \tag{3b}$$

The CM equation is uninteresting ($0 = 0$, since v_{CM} is constant). The FLT shows that the external work goes into heating the two blocks. By symmetry, half of the increase in thermal energy will appear in the upper one of the two identical blocks. Calorimetry would show an increase of $fd/2$ in the thermal energy of the upper block (assuming all the work eventually ends up as thermal agitation).

Notice that the differences between the CM and FLT equation are due to the fact that the two-block system is deformable. The work done by each individual force involves the individual displacement of the point of contact of each force. The force on the upper block moves through a distance d and does an amount of work fd, whereas the force on the lower block moves through zero distance and does zero work. In the CM equation, however, each force is multiplied by the *same* center-of-mass displacement. Similar differences can occur with rotating systems. For example, consider a cylinder rolling without slipping down an incline. The frictional force appears in the CM equation but not in the FLT, since the point of contact does not move. Equivalently, one can say that the two equations can differ if the system has internal structure with the possibility of changes of internal energy (brought about by forces which deform or rotate the system, or by other kinds of energy transfer). See Sec. X for additional discussion of this point.

Next choose just the upper block as the system (Fig. 2). To be very specific, we choose as the system of interest all those atoms pertaining to the upper block, and for simplicity we assume none of these atoms rub off and separate from the block. This is a legitimate choice of system by the usual rules for handling freebody diagrams. The equations for the upper block alone are

$$\text{CM:}\quad (f-f)\,d = \Delta\left(\tfrac{1}{2}mv_{\text{CM}}^2\right) = 0, \tag{4a}$$

$$\text{FLT:}\quad fd - fd_{\text{eff}} = \Delta E_{\text{thermal, upper block}} = fd/2. \tag{4b}$$

Again, the CM equation is uninteresting, since we move at constant speed.

The FLT, however, is most intriguing. The force f to the right does an amount of work fd. The frictional force to the left does an amount of work $-fd_{\text{eff}}$, and we will find a surprise in the effective displacement d_{eff} through which the frictional force works. The net work goes solely into increasing the thermal energy of the upper block, and we know this increase is $fd/2$, since we found that the total thermal energy rise in the two blocks together is fd. There is no net heat transfer into the upper block because the two blocks are identical: By symmetry any heat transfer from lower to upper is equal to the heat transfer from upper to lower.

If we solve the FLT for the effective displacement through which the frictional force works, we find

$$d_{\text{eff}} = d/2 \quad (!!) \tag{5}$$

How can this be? This unexpected result can be understood in the context of the standard theory of dry friction,[4,5] although the calculation in this highly symmetrical situation is not model-dependent.

IV. DRY FRICTION

When a metal block slides on a metal surface, the block is supported by as few as three protruding "teeth" (called "asperities" in the literature on friction). The very high load per unit area on these teeth causes plastic yield, and high local temperatures produced during sliding lead to adhesion (welding) in the contact regions. The frictional force divided by the tiny contact area corresponds to the large shear stress required to break these welds. This shear-

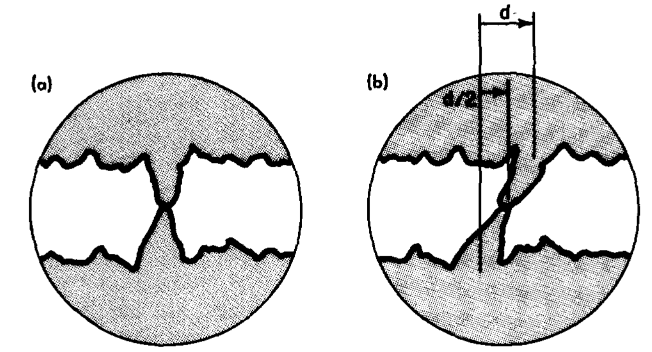

Fig. 3. (a) A stylized microscopic view of friction, with a greatly exaggerated vertical scale. Teeth belonging to the upper and lower blocks have welded together at their contact point. The shear strength of the contact is the origin of the frictional force. (b) The upper block is pulled to the right a distance d, but the contact point only moves a distance $d/2$. The frictional force f does an amount of work $-fd/2$.

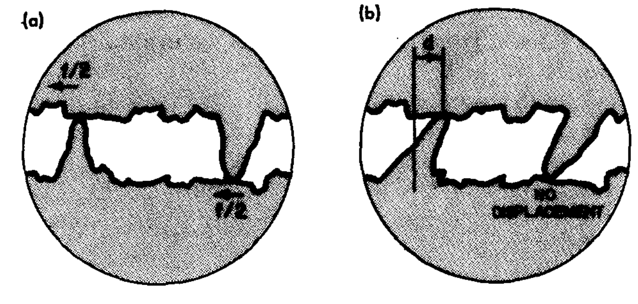

Fig. 4. (a) A more realistic picture of friction, in which the two teeth shown represent time and space averages of the contacts between the identical upper and lower blocks. The frictional force f is divided on the average into $f/2$ applied at the upper contacts and $f/2$ at the lower contacts. The vertical scale is greatly exaggerated. (b) The upper block is pulled to the right a distance d. The upper contact point also moves a distance d, but the lower contact point does not move. Work done on the upper block by the lower block is $-(f/2)(d) - (f/2)(0) = -fd/2$.

ing of contact welds is the dominant friction mechanism for a dry metal sliding on the same metal.

Because the tooth tips can become stronger than the bulk metal due to work-hardening, shearing often occurs in the weaker regions of the teeth, away from the tip. This is a major effect when the two objects are made of the same material, and chunks of metal break off and embed in the other surface. Nevertheless, this wear will be ignored in the discussion. It is in any case a symmetrical effect for identical blocks. If the metal surfaces have oxide coatings, this can reduce the shear stress required to break the temporary weld (which can reduce the coefficient of friction) and can prevent the breaking off of chunks of metal, if the oxide contact area is the weakest section.

Having briefly summarized the model of dry friction developed by Bowden and Tabor,[4,5] we proceed to use this model to calculate the work done by frictional forces exerted at the contact points.

Figure 3(a) shows in a stylized way two teeth which have temporarily adhered to each other. The vertical scale has been greatly exaggerated for clarity: machined surfaces have much gentler slopes. In Fig. 3(b) we see that when the upper block has moved a distance d to the right, the point of contact has moved a distance of only $d/2$, because the teeth of the two identical blocks are made of the same material. The time average of the contact forces is indeed f (as indicated by $f - f = Ma_{\rm CM}$), but the effective displacement is only half the displacement of the center of mass of the upper block.

Once the weld has broken the teeth can vibrate. The vibration of the tooth belonging to the upper block and thermal conduction upward from the hot tip into the main body of the block contribute to the $fd/2$ increase in the thermal energy of the upper block. Similarly, vibration and thermal conduction in the tooth belonging to the lower block end up as $fd/2$ increase in the thermal energy of the lower block.

Having two long teeth in contact as shown in Fig. 3 is rather unphysical. A better picture is that of Fig. 4(a) (again with a greatly exaggerated vertical scale), where we show one of the upper block's longest teeth in contact with the average surface level of the lower block, and one of the lower block's longest teeth similarly in contact with the average surface level of the upper block. The two teeth shown here are to be taken as representative of time and space averages of the sliding friction. The frictional force f exerted on the upper block is divided on the average into two forces each of magnitude $f/2$ at the ends of the two sets of long teeth. In Fig. 4(b) we see that when the upper block moves a distance d to the right the upper contact also moves a distance d, whereas the lower contact does not move at all. The frictional work is therefore $(-f/2)(0) + (-f/2)(d)$, which is $-fd/2 = -fd_{\rm eff}$, so $d_{\rm eff}$ is again $d/2$.

A common physical error in the treatment of this and similar situations is to mistake the CM equation for the upper block, $fd - fd = 0$, with the first law of thermodynamics, $fd - fd_{\rm eff} = fd/2$. The basic deficiency of the CM equation here is that it really has little to do with work or energy. In particular, the CM equation pays no attention to thermal energy. Yet everyone knows that the blocks get hot, and one wonders why this physical fact is not reflected in the equations. Notice that the CM and FLT equations differ here precisely because the blocks are not point particles but are deformable, so that the various forces do not share the same displacements at their points of application. In the CM equation all forces are multiplied by the *same* center-of-mass displacement, but in the FLT each force contributes an amount of work proportional to the displacement of its *own* point of application.

For contrast it is instructive to carry out the analysis in the center-of-mass frame of the two blocks, instead of the frame of the lower block. The point of frictional contact on the average is stationary in this center-of-mass frame, so $d_{\rm eff} = 0$. The external forces both act through a distance $d/2$. Considering the two-block system, each external force f does an amount of work $fd/2$, and the total work fd equals the thermal energy rise of the two blocks. If we consider just the upper block, the external force f does an amount of work $fd/2$, and the frictional force does zero work (because the contact point does not move). The total work $fd/2$ equals the thermal energy rise $fd/2$ of the upper block.

It might be expected that in general $d_{\rm eff}$ should be less than the center-of-mass displacement of the block. However, if the upper block is analyzed in its own rest frame, the frictional force exerted by the lower block acts through a nonzero distance ($d/2$) while the upper block does not move at all. A more interesting case is that of a block which accelerates from rest on a conveyor belt. Until the belt speed is reached, the sliding-friction force does *positive* work on the block, and it can be shown that the frictional force acts through a distance greater than the displacement of the block. The explanation for this odd effect is that the contact point of the friction force must move forward ahead of the block's motion, in order to accelerate the block.

V. LUBRICATED FRICTION

For the highly symmetrical process involving two blocks, the result $d_{\text{eff}} = d/2$ was independent of the particular model of friction. Nevertheless it is interesting to see that the same result is obtained in the case of lubricated sliding. Figure 5 shows the two blocks separated by a film of viscous lubricating oil, so that the two blocks do not make direct contact. Fluid layers immediately adjacent to the blocks are constrained to share the motion of the blocks. For laminar Couette flow,[6] the displacement profile in the oil is linear.

For symmetry (which eliminates the possibility of net heat transfer across the system boundary), choose as the upper system the upper block plus the upper half of the lubricating oil. It can be seen that the shear force at the midplane of the film acts through a distance which is half the displacement of the upper block: $d_{\text{eff}} = d/2$. This is the same result we obtained in our model-independent calculation and for the case of symmetrical dry friction (in the framework of the Bowden–Tabor model). Of course, the magnitude of the frictional force is much reduced by the lubrication, and the applied force f must consequently be much smaller if there is to be no acceleration. If we perform the analysis in the center-of-mass frame of the two blocks, the midplane of the oil is stationary. The upper block moves a distance $d/2$, and the applied force f does an amount of work $fd/2$, while the frictional force does no work (since the midplane does not move; $d_{\text{eff}} = 0$). The total work done on the upper system is $fd/2$, which is equal to its thermal energy rise.

VI. HEAT TRANSFER

In the case of lubricated friction it is possible to understand heat transfer in detail. Take the two-block situation, with oil between the two identical blocks. To simplify, let the thermal conductivity of the metal blocks be very high compared to the oil, so that the temperature is nearly uniform throughout a block (and rising as the process continues). Also assume the mass of the blocks is very large compared to the mass of the oil, so that almost none of the total energy rise fd is in the oil—it is all in the blocks ($fd/2$ in each). We will see how the work done on the upper system by friction (W_{fric}) and the heat transfer into the upper system (Q) depend on y, the position above the midplane where we choose to draw the system boundary. Let the reference frame be the center of mass of the two-block system. The symmetrical case, $y = 0$, is characterized by $W_{\text{fric}} = 0$ and $Q = 0$. The upper block moves a distance $d/2$, and the applied force f does an amount of work $fd/2$. The total energy transfer into the system is $fd/2$, leading to a thermal energy rise $fd/2$ in the upper block.

Figure 6(a) shows the temperature T as a function of y,

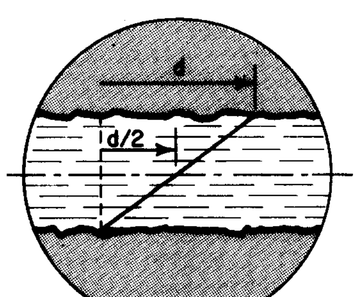

Fig. 5. A viscous lubricating oil lies between the two blocks. The displacement profile for Couette laminar flow is shown with the lower block stationary. Consider a system composed of the upper block and the upper half of the lubricating oil. The fluid shear acting on the bottom of this system acts through a distance which is half the distance through which the upper block moves.

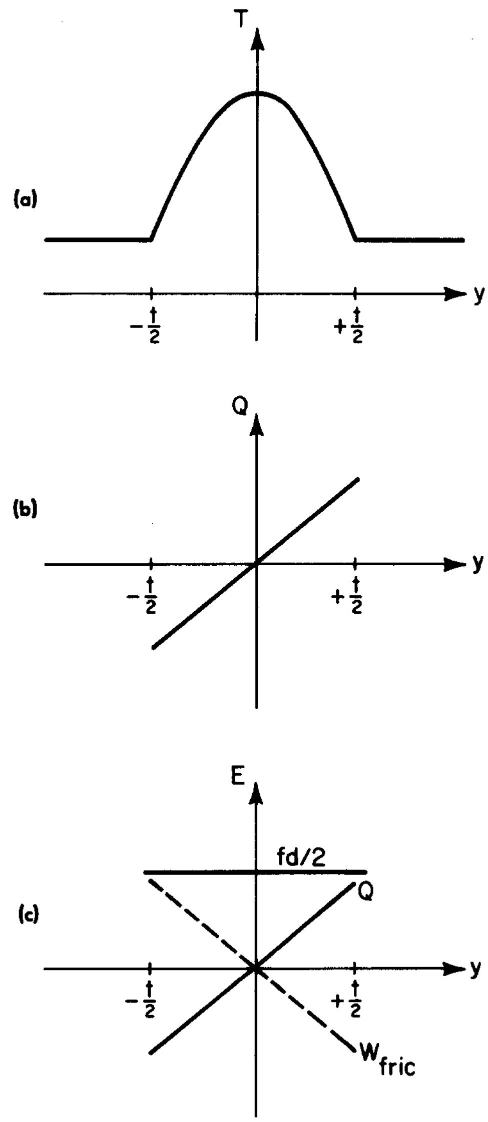

Fig. 6. The temperature (a), heat transfer into the upper system (b), and all energy transfers into the upper system (c) as a function of y, the location of the system boundary above the midplane of the oil shown in Fig. 5. Note that since the heat transfer Q is proportional to the negative gradient of the temperature, the graph in (b) is simply the negative derivative of the graph in (a).

with the highest temperature at the midplane. Figure 6(b) shows the heat transfer $Q = (-kA\partial T/\partial y) \times$ (time interval) as a function of y, the location of the system boundary. (The thickness of the oil is t, so $-t/2 < y < +t/2$.) In the simple case of uniform shear, there is uniform dissipation throughout the fluid, so that the heat-transfer graph [Fig. 6(b)] is a straight line, and the temperature graph [Fig. 6(a)] is a parabola. In any case, since Q is proportional to the negative gradient of the temperature T, the curve in Fig. 6(b) will be the negative derivative of the temperature curve shown in Fig. 6(a).

Note that at the midplane $\partial T/\partial y$ is zero, and heat flow is zero, corresponding to the symmetrical situation. For $y > 0$ heat transfer is positive, corresponding to flow into the upper system. For $y < 0$ heat transfer is negative, reflecting the fact that heat transfer flows out of the upper system into the lower oil and block.

Figure 6(c) shows all energy inputs into the upper system as a function of y, the position of the system boundary. At $y = +t/2$, $W_{\text{fric}} = -fd/2$ because the frictional force f acts to the left and the displacement $d/2$ is to the right. Q at $y = +t/2$ must be $fd/2$, since the source of the fd thermal energy rise in the two blocks is in the oil, where the energy dissipation fd occurs. At $y = -t/2$, the frictional force f acts to the left and the displacement $d/2$ is to the left, so $W_{\text{fric}} = +fd/2$ and $Q = -fd/2$ out of the upper system. The FLT for several choices of y takes these forms:

$$fd/2 + W_{\text{fric}} + Q = \Delta E_{\text{upper system}}, \quad (6a)$$

$$y = +t/2: \quad fd/2 + (-fd/2) + (+fd/2) = fd/2, \quad (6b)$$

$$y = 0: \quad fd/2 + 0 + 0 = fd/2, \quad (6c)$$

$$y = -t/2: \quad fd/2 + (+fd/2) + (-fd/2) = fd/2. \quad (6d)$$

Note that $(W_{\text{fric}} + Q)$ is zero for any choice of the system boundary y. Work and heat transfer have complementary roles in this process. Also note that since W_{fric} is just f times the displacement at the system boundary, the shape of the W_{fric} curve (and therefore also that of Q) is the same as that of the fluid displacement (a straight line in the specific case considered here).

VII. DIFFERING MATERIALS

It is instructive to see what happens in the case of dry friction if the two blocks are made of differing materials. If the friction mechanism involves making and breaking of welds, the distance d_{eff} through which the friction force acts can have any value between 0 and d (in the reference frame of the lower block), because the amount of movement of the contact point now depends on the relative stiffness of the two teeth. Also, since the situation is now asymmetrical, there may be a net heat transfer by thermal conduction from one block to the other.

If the upper block is very hard and the lower block very soft (Fig. 7), the dominant mode of friction can be "plowing,"[5] in which the hard teeth sink into the soft material and plow through it. The frictional force exerted by the soft material on the hard teeth acts through the same distance as the motion of the upper block: $d_{\text{eff}} = d$, and the work done by the frictional force is $-fd$. The FLT for the upper block is $fd - fd = \Delta E_{\text{upper}} = 0$, implying that the hard upper block does not get hot.

However, if the soft lower block becomes hotter than the upper block, some net heat transfer by thermal conduction of amount Q across the system boundary can occur, out of the lower block into the upper block. The work performed on the upper block is $fd - fd = 0$, while the work performed on the lower block is fd. This makes the FLT for the several systems appear as

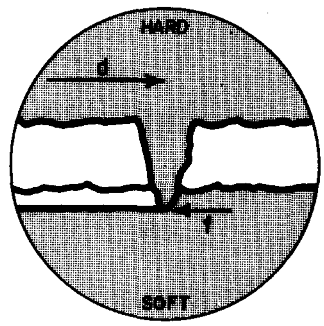

Fig. 7. If the upper block is very hard and the lower block very soft, the teeth of the upper block may plow through the soft material. The displacement of the point of application of the frictional force is d, and the frictional force does an amount of work $-fd$ on the upper block.

$$Q = \Delta E_{\text{upper}} \quad (7a)$$

$$fd - Q = \Delta E_{\text{lower}} \quad (7b)$$

$$\overline{fd = \Delta E_{\text{upper + lower}}}. \quad (7c)$$

In the asymmetrical case it is generally not possible to identify just the heat transfer term alone. In the case of lubricated friction we were able to show explicitly the complementary roles of work and heat transfer, but for dry friction this is difficult to do. This difficulty does not affect the calculations in the case of identical blocks, since then $Q = 0$ by symmetry.

At the other extreme, if the upper block is very soft and the lower block very hard, the soft upper block is dragged across stationary hard teeth (Fig. 8). In this case the contact point does not move, and $d_{\text{eff}} = 0$. The frictional force does no work on the upper block. For the upper block we have $(f)(d) - (f)(0) = fd = \Delta E_{\text{upper}}$, and the soft upper block gets hot. Again, there can be some net heat transfer from the hotter upper block.

It is worth mentioning here that Tabor[5] objects very much to our common usage of the words "rough" and "smooth" to refer to surfaces with or without friction. He points out that the modern understanding of friction shows that the degree of surface roughness need not have much correlation with frictional phenomena.

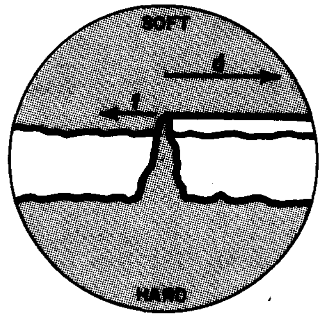

Fig. 8. If the upper block is very soft and the lower block very hard, the soft upper block is dragged across stationary teeth. The frictional force acts through zero distance and does no work on the upper block.

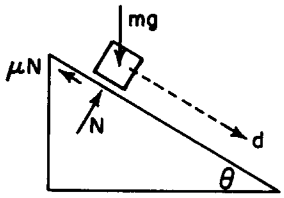

Fig. 9. A block slides down an incline with friction. The work done by the frictional force is *not* $-\mu Nd$.

VIII. TEXTBOOK EXAMPLE

A favorite problem in textbooks has a block sliding a distance d down an incline with friction (Fig. 9). In this case the block's hot teeth are continually encountering new, cold regions of the incline, so that it is likely that there will be net heat transfer $|Q|$ out of the block into the incline. (In the two-block situation, there was symmetry in such contacts.) Taking the block as the system of interest, the relevant equations are

$$\text{CM:}\quad (mg\sin\theta - \mu N)d = \Delta\left(\tfrac{1}{2}mv^2\right), \tag{8a}$$

$$\text{FLT:}\quad (mg\sin\theta)d - \mu N d_{\text{eff}} - |Q| = \Delta\left(\tfrac{1}{2}mv^2\right) + \Delta E_{\text{thermal of block}}. \tag{8b}$$

Students are often asked to calculate "the work done by the frictional force." However, since this quantity depends on the unknown effective displacement d_{eff} of the frictional force, the frictional work cannot be calculated. What is really being asked for is the term $-\mu Nd$, whereas the actual work done by the frictional force is only $-\mu N d_{\text{eff}}$.

A comparison of the CM and FLT equations yields

$$\mu N(d - d_{\text{eff}}) = \Delta E_{\text{thermal of the block}} + |Q|. \tag{9}$$

Since the right side of this relation is positive, this indicates d_{eff} will be less than d.

If we consider the universe (block, incline, and earth), we have for the FLT

$$0 = \Delta\left(\tfrac{1}{2}mv^2\right) - (mg\sin\theta)d + \Delta E_{\text{thermal of block}} + \Delta E_{\text{thermal of inlcline}}. \tag{10}$$

Comparing with the FLT for the block alone, we see that

$$\Delta E_{\text{thermal of the block}} = \mu N(d - d_{\text{eff}}) - |Q| \tag{11a}$$

$$\Delta E_{\text{thermal of the incline}} = \mu N d_{\text{eff}} + |Q| \tag{11b}$$

$$\Delta E_{\text{thermal of the universe}} = \mu Nd. \tag{11c}$$

IX. ROTATIONAL FRICTION

There is a rotational analog of the CM equation. Starting from the torque equation about the center of mass,

$$\sum_i \tau_{i_{\text{CM}}} = I_{\text{CM}}\alpha, \tag{12}$$

and integrating through a rotation θ one obtains the "rotational CM equation,"

$$\int\left(\sum_i \tau_{i_{\text{CM}}}\, d\theta\right) = \int I_{\text{CM}}\frac{d\omega}{dt}\, d\theta = \Delta\left(\tfrac{1}{2}I_{\text{CM}}\omega^2\right). \tag{13}$$

Note that this is *not* an energy equation. It leaves no room for thermal energy terms. The *same* rotation angle θ has been used for all the torques, whereas work done by external torques is

$$W = \sum_i\left(\int \tau_{i_{\text{CM}}}\, d\theta_i\right), \tag{14}$$

and different torques $\tau_{i_{\text{CM}}}$ may act through different rotational angles θ_i.

Consider a wheel being rotated by an external motor on an axle with friction. If τ_m = the torque applied by the motor, τ_a = the frictional torque applied by the axle, and $|Q|$ = the net heat transfer from the wheel to its surroundings (including the axle), we have

$$\text{CM:}\quad (\tau_m - \tau_a)\theta = \Delta\left(\tfrac{1}{2}I_{\text{CM}}\omega^2\right), \tag{15a}$$

$$\text{FLT:}\quad \tau_m\theta - \tau_a\theta_{\text{eff}} - |Q| = \Delta\left(\tfrac{1}{2}I_{\text{CM}}\omega^2\right) + \Delta E_{\text{thermal of wheel}}. \tag{15b}$$

Due to deformation of the "teeth" where the wheel contacts the axle, the effective (angular) displacement θ_{eff} of the axle torque is less than the angular displacement θ of the wheel as a whole. We find by comparing the CM and FLT equations that

$$\tau_a(\theta - \theta_{\text{eff}}) = \Delta E_{\text{thermal of wheel}} + |Q|. \tag{16}$$

Note that a truly rigid wheel (all θ_i's the same) *cannot* account for friction.

X. THE IFLT, AN INVARIANT FORM OF THE FLT

Although the CM equation is correct in any inertial frame, the numerical values of the various terms in the equation vary from frame to frame, since the displacement and velocity of the center of mass of the system obviously depend on the choice of reference frame. These noninvariant terms also appear in the FLT. However, subtracting the CM equation from the FLT yields a version of the FLT whose terms *are* invariant. This equation might be called the invariant FLT, or IFLT. Let the point of application of the ith external force be $\mathbf{r}_{\text{CM}} + \mathbf{R}_i$, so that $\mathbf{R}_i$ is the location of this point relative to the center of mass. Then we have

$$\text{FLT:}\quad \sum_i\int \mathbf{F}_{i_{\text{external}}}\cdot(d\mathbf{r}_{\text{CM}} + d\mathbf{R}_i) + Q = \Delta\left(\tfrac{1}{2}Mv_{\text{CM}}^2\right) + \Delta E_{\text{internal}}, \tag{17}$$

$$\text{CM:}\quad \sum_i\int \mathbf{F}_{i_{\text{external}}}\cdot d\mathbf{r}_{\text{CM}} = \Delta\left(\tfrac{1}{2}Mv_{\text{CM}}^2\right), \tag{18}$$

$$\text{IFLT:}\quad \sum_i\int \mathbf{F}_{i_{\text{external}}}\cdot d\mathbf{R}_i + Q = \Delta E_{\text{internal}}. \tag{19}$$

$\Delta E_{\text{internal}}$ represents both macroscopic and microscopic energy in the rest frame of the system, and can be thought of as the rest energy of the system. We used the IFLT implicitly several times in this paper when we subtracted the CM equation from the FLT to get at the interesting energetics of a process. This induced us to identify the IFLT as being of general interest.

In the IFLT the force integral involves only displacements relative to the center-of-mass point, so this integral is clearly independent of our choice of reference frame. Since the internal energy change (or change in rest energy) is also frame-independent, the IFLT shows that the heat transfer Q is invariant. (Q may be considered a cover term for heat transfer plus other nonwork energy transfers across the system boundary, such as radiation or mass transfer.) The form of the IFLT makes it easy to see that the FLT and CM equations in the absence of heat transfer differ only if there is deformation or rotation or, equivalently, if the internal energy can change (implying an internal structure). One may think of the IFLT as the energy equation in the center-of-mass frame, and it is valid even if the center of mass accelerates.

Note that the IFLT relates meaningful energy transfers across the system boundary to significant changes in the internal energy of the system. In contrast, the CM equation can be considered to describe merely the "kinematics" of the center-of-mass motion, saying nothing about significant energy changes inside the system. It is interesting to note that if $\mathbf{v}_{CM,i}$ and $\mathbf{v}_{CM,f}$ are the initial and final velocities of the center of mass in some inertial frame, the CM equation reduces to $0 = 0$ in another inertial frame moving at a velocity $(\mathbf{v}_{CM,i} + \mathbf{v}_{CM,f})/2$. In this special frame the initial and final center-of-mass speeds are the same, though the direction of motion may differ.

The force integral in the FLT is of course referred to as work. It has been suggested[2] that the integral in the CM equation be called pseudowork. Perhaps we should call the force integral in the IFLT the "invariant work." The relationship among these defined quantities is that work equals pseudowork plus invariant work.

The IFLT offers a useful perspective on energy transfers. For example, the invariant work performed by a frictional force on a single block is positive, since the frictional force acts in the same direction as the bending of a tooth (relative to the center of mass). This positive invariant work is associated with an increase in the internal energy of the block. Depending on reference frame, the work done on a single block by frictional forces in the FLT may be positive or negative, but the corresponding invariant work is positive, independent of frame. The IFLT, and the geometry of the frictional contact point, can be used to generalize the results of Eqs. (7c) and (11c), to show that the total internal energy rise in two blocks sliding on each other is equal to the absolute value of the product of the frictional force times the relative displacement of the two center-of-mass points.

As another example, consider a skater pushing off from a wall. The force exerted by the wall on the skater's hand does no work on the skater, because there is no displacement of the point of application. In the FLT, the skater's increased kinetic energy comes at the expense of internal energy.[7] The wall force does produce pseudowork, since the center of mass of the skater moves, and in the CM equation this positive pseudowork is equal to the kinetic energy increase. The wall force produces *negative* invariant work, because the point of application of the force moves away from the center of mass. In the IFLT this negative invariant work is equal to the decrease in the skater's internal energy (with a change of sign, one can say that the skater does positive invariant work on the surroundings). Here are several quite different perspectives on the same process. It is a useful exercise to apply the IFLT to the various problems treated in Ref. 3.

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For those readers who may have seen a preprint of an earlier version of this article with Sherwood as sole author, we should explain that during the submission process it was discovered that both authors had independently reached similar conclusions and had submitted closely related papers. As a result, we decided to publish this article as co-authors.

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